Mean Field Convergence of a model of multiple TCP connections through a buffer implementing RED

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Mean Field Convergence of a model of multiple TCP connections through a buffer implementing RED

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Abstract: RED (Random Early Detection) have been suggested when multiple TCP sessions are multiplexed through a bottleneck buffer. The idea is to detect congestion before the buffer overflows by dropping or marking packets with a probability that increases with the queue length. The objectives are an equitable distribution of packet loss, reduced synchronization together with reduced packet loss, delay, and delay variation.

Baccelli, McDonald and Reynier [1] have proposed a model for multiple TCP connections in the congestion avoidance regime multiplexed through a bottleneck buffer implementing RED. The window sizes of each TCP session evolve like independent dynamical systems coupled by the queue length at the buffer. The key idea in [1] is to consider the histogram of window sizes as a random measure coupled with the queue. Here we prove the conjecture made in [1] that as the number of connections tends to infinity this system converges to a deterministic mean-field limit comprising the window size density coupled with a deterministic queue.

Key-words: TCP, RED, mean-field, dynamical systems.

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Convergence en champ moyen de sources TCP qui partagent une file d'attente munie de RED

Résumé : L'algorithme RED a été suggéré lorsque plusieurs sessions TCP sont en concurrence dans un routeur-goulet d'étranglement. L'idée est de détecter la congestion avant d'avoir rempli la file d'attente en détruisant ou en marquant les paquets avec une probabilité qui croît avec la longueur de la file d'attente. Les objectifs sont de distribuer équitablement les pertes de paquets, de réduire la synchronisation entre les sources TCP et de réduire les délais et leurs variations. Baccelli, McDonald and Reynier [1] ont proposé un modèle des connexions TCP dans le régime dit de "Congestion Avoidance" (évitement de congestion) qui sont en concurrence dans un routeur qu'elles congestionnent. Ce routeur utilise RED. Les tailles des fenêtres de chaque connexion TCP sont des systèmes dynamiques qui évoluent indépendamment avec comme couplage la taille de la file d'attente. L'idée centrale dans [1] est de considérer l'histogramme des tailles de fenêtre comme une mesure aléatoire couplée avec la file d'attente. Dans le présent article nous démontrons la conjecture faite dans [1] qui dit que lorsque le nombre de connexions TCP tend vers l'infini, ce système converge vers une limite du champ moyen qui est déterministe. Cette limite est un système comprenant une densité de tailles de fenêtres et une taille de file d'attente déterministe.

Mots-clés : TCP, RED, Champ moyen, systèmes dynamiques.

1 Introduction

Imagine the scenario where N work stations in a university department are connected by a switched ethernet to a departmental router. If every work station simultaneously FTP's a file to some distant machine then the output buffer in the router will be a bottleneck. We study the interaction of N TCP/IP connections in the congestion avoidance phase of TCP Reno routed through a bottleneck queue.

Upon receiving a packet the recipient sends back an acknowledgement packet so there is one Round Trip Time (RTT) between the time a packet is sent and the acknowledgement is received. The acknowledgement contains the sequence number of the highest value in the byte stream successfully received up to this point in time. When packets arrive out of order or are lost then duplicate acknowledgements are generated. Each source implements a window flow control which limits the number of packets from this connection allowed into the network during one Round Trip Time (RTT).

The link rate of the router is NL packets per second. We assume the packets from all connections join the queue at the bottleneck buffer and we denote by $Q_N(t)$ the average queue per flow. We assume the scheduling to be FIFO.

We imagine the source writes its current window size and the current RTT in each packet it sends where by the current RTT we mean the RTT of the last acknowledged packet.

- $W_n^N(t)$ is defined to be the window size written in a packet from connection n arriving at the router at time t;
- $R_n^N(t)$ is defined to be the RTT written in a packet from connection n arriving at the router at time t (this RTT is the sum of the propagation delay plus the queueing delay in the router).
- We shall assume that connections can be divided in d classes where connections in class c have common transmission time T_c.

Under TCP Reno, established connections execute congestion avoidance where the window size of each connection increases by one packet each time a packet makes a round trip, i.e. each R_n^N as long as no losses or timeouts occur. During this phase the rate the window of connection n increases is approximately $1/R_n^N$ packets per second. The only thing restraining the growth of transmission rates is a loss or timeout. When a loss occurs the window is reduced by half.

The source detects a loss when three duplicate acknowledgements arrive. The source then starts a fast retransmit/fast recovery by immediately resending the lost packet. We assume the losses are only generated by the RED (Random Early Detection) active buffer management scheme or by tail-drop. We neglect the possibility that duplicate acknowledgements are generated by packets arriving out of order. We will assume the buffer holds B packets and that once this buffer space is exhausted arriving packets are dropped. Such tail-drops come in addition to the RED mechanism. Here we take the drop probability of RED (of an incoming packet before being processed) to be a function of the queue size which is zero for a queue length below Q_{min} but rises linearly to p_{max} at Q_{max} and equals to 1 above Q_{max}.

If all N connections are in congestion avoidance we can reformulate this drop probability in terms of Q^N, the queue size divided by N, as $F(Q^N(t))$, where F is a distribution function which is zero below $q_{min} = Q_{min}/N$ but rises linearly to p_{max} at $q_{max} = Q_{max}/N$ and jumps to 1 when $Q^N(t) \geq q_{max}$. Of course the tail-drop scheme can be considered as the limiting case when $q_{min} = 0$, $q_{max} = b$ and $p_{max} = 0$.

The goal of this paper is to prove the conjecture in [1] which claims that the histogram of the window sizes in any class c converges to a deterministic mean field limit with measure $M_c(t, dw)$ at time t and moreover the relative queue size $Q^N(t)$ converges to a deterministic limit $Q(t)$.

Theorem 1 *Under Assumptions [1]-[6] given in Section 2, as $N \to \infty$, the random measure of the window sizes of connections in each class c converge to a deterministic measure $M_c(t, dw)$ which is the marginal distribution of $M_c(s - R_c(s), dv; s, dw)$, the deterministic joint distribution of the window sizes at time t and at time $t - R_c(t)$. Let $\mathcal{G} = \{g \in C_b^1(I\!R^+) : g(0) = 0\}$ where $C_b^1(I\!R^+)$ is the space of bounded functions with bounded derivatives. For functions $g_c \in \mathcal{G}, c = 1, \ldots d$*

$$
\begin{aligned}
&\langle g_c, M_c(t)\rangle - \langle g_c, M_c(0)\rangle \\
&= \int_0^t \Big[\frac{1}{R_c(s)}\langle\frac{dg_c(w)}{dw}, M_c(s, dw)\rangle \qquad (1.1)\\
&+ \langle (g_c(w/2) - g(w))v, M_c(s - R_c(s), dv; s, dw)\rangle \frac{(1 - \dot{R}_c(s))}{R_c(s - R_c(s))} K(s - R_c(s))\Big] ds. \\
&= \int_0^t \Big[\frac{1}{R_c(s)}\langle\frac{dg_c(w)}{dw}, M_c(s, dw)\rangle \\
&+ \langle (g_c(w/2) - g(w)), e(s, s - R_c(s), w) M_c(s, dw)\rangle \frac{(1 - \dot{R}_c(s))}{R_c(s - R_c(s))} K(s - R_c(s))\Big] ds.
\end{aligned}
$$

$$= \int_0^t \Big[\frac{1}{R_c(s)}\langle\frac{dg_c(w)}{dw}, M_c(s,dw)\rangle + \frac{1}{R_c(s-R_c(s))}K(s-R_c(s))(1-\dot{R}_c(s)) \tag{1.2}$$
$$\cdot\langle g_c(w), e(s,s-R_c(s),2w)\cdot M_c(s,2dw) - e(s,s-R_c(s),w)\cdot M_c(s,dw)\rangle\Big]ds.$$

where $\langle g_c, M_c(t)\rangle = \int_{w=0}^{\infty} g_c(w)M_c(t,dw)$, *where* $R_c(t) = T_c + Q(t-R_c(t))/L$ *and where*

$$e(s,s-R_c(s),w) = \langle v, \frac{M_c(s-R_c(s),dv;s,dw)}{M_c(t,dw)}\rangle$$

is the conditional expectation of the window one RTT in the past given the window is now w.

Moreover the queue size converges to a deterministic limit $Q(t)$ *satisfying*

$$\frac{dQ(t)}{dt} = \sum_{c=1}^{d}\int_w wM_c(t,dw)\frac{(1-K(t))}{R_c(t)} - L \tag{1.3}$$
$$-\left(\sum_{c=1}^{d}\langle w, M_c(t,dw)\rangle\frac{(1-K(t))}{R_c(t)} - L\right)^+ \chi\{Q(t)=q_{max}\} \tag{1.4}$$
$$-\left(\sum_{c=1}^{d}\langle w, M_c(t,dw)\rangle\frac{(1-K(t))}{R_c(t)} - L\right)^- \chi\{Q(t)=0\}). \tag{1.5}$$

For $0 < Q(t) < q_{max}$, $K(t) = F(Q(t))$; *when* $Q(t) = q_{max}$), $K(t)$ *is determined by*

$$\sum_{c=1}^{d}\langle w, M_c(t,dw)\rangle\frac{(1-K(t))}{R_c(t)} = L.$$

Assuming $g_c(0) = 0$ and that $g_c(w) \to 0$ as $w \to \infty$ we can rewrite (1.2) as

$$\langle g_c, M_c(t)\rangle - \langle g_c, M_c(0)\rangle$$
$$= \int_0^t \Big[-\frac{1}{R_c(s)}\langle g_c(w), \frac{M_c(s,dw)}{dw}\rangle + \frac{1}{R_c(s-R_c(s))}K(s-R_c(s))(1-\dot{R}_c(s))$$
$$\cdot\langle g_c(w), e(s,s-R_c(s),2w)\cdot M_c(s,2dw) - e(s,s-R_c(s),w)\cdot M_c(s,dw)\rangle\Big]ds.$$

where $\frac{M_c(s,dw)}{dw}$ is the Frechet derivative of the measure M_c with respect to w. Consequently,

$$\frac{dM_c(t)}{dt} = -\frac{1}{R_c(t)}\frac{M_c(t,dw)}{dw} + \frac{1}{R_c(t-R_c(t))}K(t-R_c(t))(1-\dot{R}_c(s)) \quad (1.6)$$
$$\cdot (e(t,t-R_c(t),2w)(M_c(t,d(2w)) - e(t,t-R_c(t),w)M_c(t,dw))\Big].$$

In Section 3.3.2 in [1] we made a smooth approximation to $e(t,t-R_c(t),w)$ based on the fact that one RTT in the past the window size was most likely the current window size minus one or twice the current window size if a loss was detected in the interim. With this approximation we used (1.6) to evolve a discrete approximation of the measure M_c (except [1] only treats one class). The numerical results are excellent after one corrects for the fact that a proportion of the connections in an Opnet simulation are in timeout (our model assumes connections instantaneously resume congestion avoidance if they fall into timeout).

To illustrate the mean field limit we performed an Opnet simulation with $N = 200$, $N = 400$ and $N = 800$ sources. Each source sends packets of size 536 bytes to a T3 router with a transmission rate of 44.736 Megabits per second or $L = 10433$ packets per second. We assume the sources all have a transmission delay of 100 milliseconds. The router implements RED with $p_{max} = 0.05$ for all the simulations but we rescale Q_{max} to be 1000 with 200 sources, 2000 with 400 sources and 4000 with 800 sources. Since Q_{max} scales with N, the average queue size does as well while holding the loss probability fixed which in turn holds the average window size fixed. As N increases we see the fluctuations in the relative queue size (in packets per connection) decreases. We also see the relative queue size of the Matlab numerical simulation is a bit high. This is because of timeouts as discussed in [1].

Neither $M_c(t,dw)$ or $M_c(s-R_c(s),dv;s,dw)$ is a state but the above equation does provide enough information to evolve the system. Let $\mu_c(t)$ denote the process $\{M_c(s,dw); t-1 \leq s \leq t\}$ (all RTT's are less than 1). Using (1.1) we can evolve $M_c(t,dw)$ from t to $t+\delta t$ while $M_c(t-s+\delta t,dw)$ is obtained by a time shift. Unfortunately μ_c is not practical state. Even if we discretize and only keep the trajectory of the process on a partition giving $\{M_c(s_i,dw); t-1 = s_0 < s_1 \ldots s_n = t\}$ it still requires too much computer memory to solve numerically.

We can avoid this problem by defining a sequence of times t_k^c for each class such that $t_{k+n+1}^c - R_c(t_{k+n+1}^c) = t_k^c$. If we pick n sufficiently large this gives a fine partition. Define $\Phi^c(t) =$ the first k such that $t_k^c > t$. We will construct our solution by recurrence from time t_i to t_{i+1} by defining $t_{i+1} = min_c(t_{\Phi^c(t_i)}^c)$ starting from time $t_0 = 0$. Next assume that for each class we have been able to calculate and

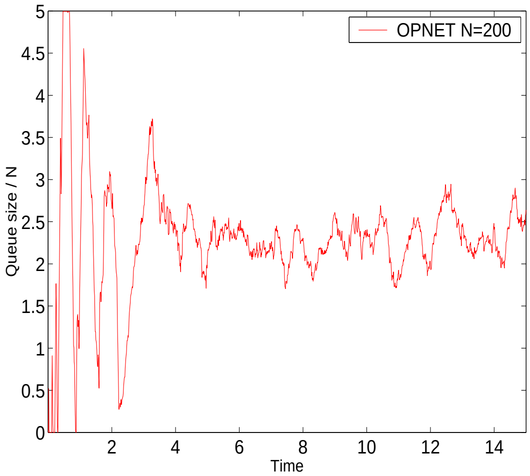

Figure 1: Relative queue size with 200 sources

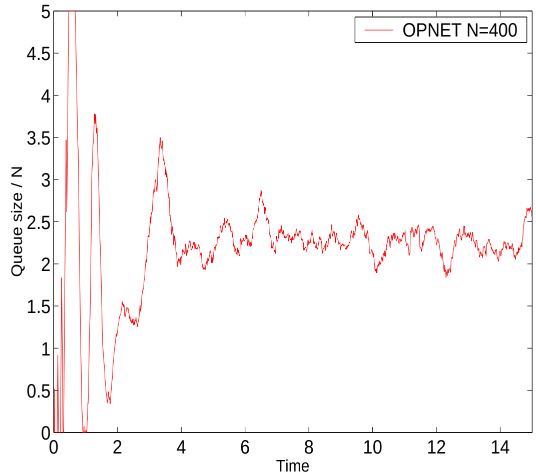

Figure 2: Relative queue size with 400 sources

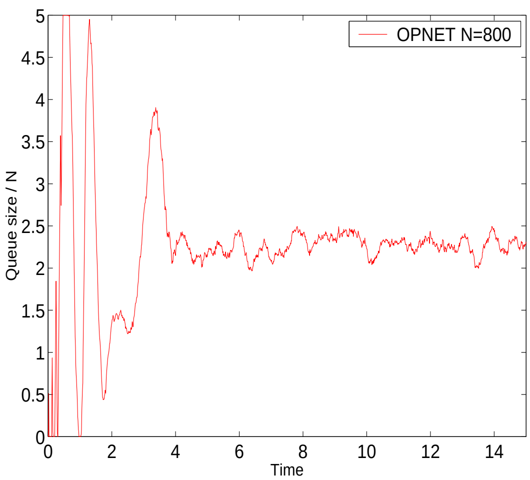

Figure 3: Relative queue size with 800 sources

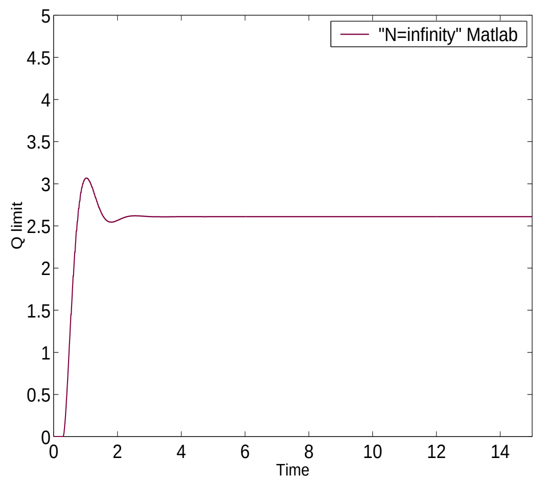

Figure 4: Queue size of the mean field limit

save the vector $V_c^M(t)$, a discretized version of $M_c(t_k^c)$ for $k = m - n, \ldots m$ where $m = \Phi^c(t) - 1$. (these are marginals not the entire joint distribution). Also assume we save the vector of kernels $V_c^T(t)$ given by $T^c(t_k^c)$ for $k = m - n, \ldots m$ where

$M_c(t^c_{k+1}) = M_c(t^c_k) \circ T^c(t^c_{k+1})$ and m is as above. Finally assume that we save the kernels $S^c_m = \prod_{k=m-n}^{m} T^c(t^c_k)$.

We can now evolve our system to t_{i+1}. At each step we only evolve the queue and the one class d where $t_{i+1} = t^d_{\Phi^d(t_i)}$. The inverse kernel $(S^d_m)^{-1}$ gives the conditional distribution of the windows of class d one RTT before time t_i given the window at time t_i. Calculate the conditional expectation $e(t_i, t_i - R_d(t_i), w)$. With this we can use (1.2) to calculate T^d_{m+1}. Drop T^d_{m-n}. We also update $S^d(m+1) = (T^d_{m-n})^{-1} S^d(m) T^d(m+1)$. Finally we calculate $Q(t_{i+1})$ using the $M_c(\Phi^c(t_i) - 1)$.

Since we are interested in a mathematical proof of the convergence to the mean field we will not try to justify the numerical procedure leading to the above results. We will also ignore timeouts and slow-start as well as special details of congestion avoidance which would only serve to obscure the main ideas. We will nevertheless sketch how these extensions could be handled.

The structure of the paper is as follows. In Section 2 we model the N particle system of windows coupled with the queue and then formulate this as a histogram of window sizes coupled with a queue. In Section 3 we summarize the mean field limit. The proof of the existence of this limit follows in Section 4. In Section 5 we establish the convergence to a unique limit. Finally in Section 6 we establish Theorem 1.

2 The N-particle system and mean-field limit

2.1 The N-particle Markov process

Our model takes into account the delay of one round trip time between the time the packet is killed and the time when the buffer receives the reduced rate. We assume window reductions at connection n occur because of losses one round trip time in the past. To first order, the probability of a window reduction between time t and $t+h$ is

$$\int_{t-R^N_n(t)}^{t+h-R^N_n(t+h)} \frac{W^N_n(s)}{RTT^N_n(s)} F(Q^N(s))ds \sim [1-\frac{d}{dt}R^N_n(t)]\frac{W(t-R^N_n(t))}{R^N_n(t-R^N_n(t))}F(Q^N(t-R^N_n(t)))h$$

since the probability a packet is dropped is proportional to $W(t-R^N_n(t))/R^N_n(t-R^N_n(t))$, the transmission rate one time in the past, times $F(Q^N(s))$, the drop rate one round trip in the past. We therefore model the process of window reductions by a Poisson process with stochastic intensity

$$\lambda^N_n(t) := [1 - \frac{d}{dt}R^N_n(t)]\frac{W^N_n(t-R^N_n(t))}{R^N_n(t-R^N_n(t))}F(Q^N(t-R^N_n(t)))$$

(we can assume $W_n^N(t) = w_n(0)$ for $t < 0$). The term $[1 - \frac{d}{dt}R_n^N(t)]$ in λ_n^N was overlooked in [1].

Let $\{N_n(t); n = 1, \ldots\}$ be independent Poisson processes with intensity 1 and let $\Lambda_n^N(t) = \int_0^t \lambda_n^N(s)ds$ be the stochastic intensity for the Poisson point process of losses of connection n. Hence the losses of connection n occur according to the time changed Poisson process $N_n(\Lambda_n^N(t))$.

Below we will make a fluid approximation of the the queue and the window sizes thus ignoring fluctuations at the packet level. This allows us to describe the windows as independent differentiable dynamical systems coupled with a fluid buffer.

Differential equation for windows: There are three separate phases: congestion avoidance, timeout and slow start. During congestion avoidance, when no loss occurs the window size increases linearly at rate $1/R_n^N(t)$ but if the source detects the loss of a packet at time $t - R_n^N(t)$ because three duplicate acknowledgements arrive, the source cuts the current $W_n^N(t^-)$ size by half to $W_n^N(t^-)/2$. The slow start threshhold (ssthresh) is set to $H_n^N(t) = W_n^N(t^-)/2$.

The source then begins fast retransmit and fast recovery. The lost packet is retransmitted and through window inflation packets continue to be sent as if the window size is constant (or at least the average transmission rate is consistent with a constant window size $W_n^N(t^-)/2$). When the retransmitted packet is acknowledged congestion avoidance resumes. Hence the evolution of the window size in the congestion avoidance phase is described by the following stochastic differential equation:

$$dW_n^N(t) = \frac{1}{R_n^N(t)}(1 - S_n^N(t))dt - \frac{W_n^N(t^-)}{2}dN_n(\Lambda_n^N(t)),$$

with $W_n^N(0) = w_n(0)$, $n = 1, \ldots, N$ specified. Here $S_n^N(t) = 1$ if this connection is still in fast recovery at time t and is zero otherwise.

If $W_n^N(t^-) < 4$ then the source can't receive three duplicate ACKs so it will go into the timeout phase for approximately one second. Other events like the loss of the retransmitted packet will also provoke a timeout. If we wished to model timeouts we could define a function $U(W_n^N(t^-))$ equal to one if the connections falls into timeout during fast recovery and zero if not. Hence the point process of falling into timeout is given by $U(W_n^N(t^-))dN_n(\Lambda_n^N(t))$. During the timeout phase the window size is zero. The connection is described by ssthresh and the remaining time in timeout.

After the timeout phase elapses the source enters slow-start and doubles its window size starting from one every RTT until the window size reaches ssthresh at which time congestion avoidance restarts. If another loss is detected before reaching the congestion avoidance phase, the connection will go into timeout. During the slow-start phase the connection is described by the window size and ssthresh.

At any time t a certain proportion of the N connections will be in each phase. In the mean-field limit these proportions will converge to deterministic fractions. We will not show this here. In fact we will simply ignore all the special details of fast recovery and timeouts so we omit the factors $S_n^N(t)$ and $U(W_n^N(t^-))$. Consequently we assume the windows evolve as

$$dW_n^N(t) \quad = \quad \frac{1}{R_n^N(t)}dt - \frac{W_n^N(t^-)}{2}dN_n(\Lambda_n^N(t)), \tag{2.7}$$

with $W_n^N(0) = w_n(0)$, $n = 1, \ldots, N$ specified.

Assumptions on the initial state:

Assumption 1 *1) Prior to time zero the window size of connection n is a constant $w_n(0)$ and $0 \leq w_n(0) \leq W_{max}$.*

2) $\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \frac{1}{w_n(0)^2} \leq \underline{\mu}(0)$. This is just a technical assumption since in we don't really start in congestion avoidance but rather with in slow-start.

3) The transmission time T_n satisfies $T_{min} \leq T_n \leq T_{max}$ for all n.

4) $Q^N(0) = q(0)$ a constant.

Bound $a(t)$ for the window size at time t: From Assumption 1),

$$a(t) := W_{max} + \frac{t}{T_{min}} \geq w_n(0) + \frac{t}{T_n} \geq W_n^N(t)$$

at every time t . The stochastic intensity for the Poisson point process of losses of connection n is $\lambda_n^N(s) \leq a(s)/T_{min} =: \overline{\lambda}(t)$ for all $0 \leq t \leq T$. Consequently $\Lambda_n^N(t) \leq \overline{\Lambda}(t)$ where $\overline{\Lambda}(t) = \int_0^t \overline{\lambda}(s)ds$.

Differential equation for queue size: We assume packets have equal mean sizes of 1 data unit. We also assume that the instantaneous window and throughtput are linked by a Little type formula. Hence, for $Q(t) < q_{max}$, $K^N(t) = F(Q^N(t))$ and the rate of change of the fluid buffer is given by

$$
\begin{aligned}
N\frac{dQ^N(t)}{dt} &= \sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - NL \\
&\quad - \left(\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - NL\right)^{+}\chi\{Q^N(t)=q_{max}\} \\
&\quad + \left(\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - NL\right)^{-}\chi\{Q^N(t)=0\}
\end{aligned}
$$

since the proportion $K^N(t) := F(Q^N(t))$ of the fluid is lost. The second term prevents the queue from exceeding q_{max} while the third prevents the queue size from becoming negative. In effect the queue can stick at 0 until a sufficient number of connections increase their window size.

Dividing by N gives

$$
\begin{aligned}
\frac{dQ^N(t)}{dt} &= \frac{1}{N}\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - L \qquad (2.8)\\
&\quad - \left(\frac{1}{N}\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - L\right)^{+}\chi\{Q^N(t)=q_{max}\} \qquad (2.9)\\
&\quad + \left(\frac{1}{N}\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1-K^N(t)) - L\right)^{-}\chi\{Q^N(t)=0\},
\end{aligned}
$$

with $Q^N(0) = q(0)$.

If $Q^N(t)$ were to exceed q_{max} then the loss rate is 100% so in fact the queue jitters at q_{max} and while $Q^N(t) = q_{max}$ the loss rate is $K^N(t)$ where

$$
(1-K^N(t))\frac{1}{N}\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)} = L
$$

Note that $(1-K^N(t)) \geq 1-p_{max}$ as long as $Q^N(t) < q_{max}$. If $Q^N(t) = q_{max}$ then $(1-K^N(t)) \geq LT_{min}/a(t)$. Either way we have $(1-K^N(t)) \geq (1-k_{max}) > 0$.

Relation between RTT and queue size: Because of the FIFO assumption, R_n^N, the RTT of connection n, should satisfy

$$
R_n^N(t) = T_n + Q^N(t - R_n^N(t))/L, \qquad (2.10)
$$

where T_n is the propagation delay from source n to the destination and back. In this case all the delay is after the router but one can imagine other scenarios which would alter this equation.

We could define $\phi_n^N(s)$ to be the future round trip time written into a packet leaving the source at time s. For the above scenario $\phi_n^N(s) = T_n + Q^N(s)/L$. Note that if $s + \phi_n^N(s) = t$ then $R_n^N(t) = t - s$. Also note that $s + \phi_n^N(s)$ is monotonic (and hence R_n^N is well defined) because the derivative, if $Q^N(t) < q_{max}$, is

$$\begin{aligned} 1 + \frac{1}{L}\frac{dQ^N(s)}{ds} &= 1 + \frac{1}{L}\left(\frac{1}{N}\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1 - K^N(t)) - L\right) \\ &\geq \frac{1}{L}\frac{1}{N}\sum_{n=1}^{N} W_n^N(t)\frac{1}{T_{min}}(1 - k_{max}). \end{aligned}$$

This is positive unless all the window sizes are zero and this has probability zero. If $Q^N(t) = q_{max}$ then the derivative is one.

Note that by taking the derivative of (2.10) we get

$$(1 - \dot{R}_n^N(t)) = \frac{1}{1 + \dot{Q}^N(t - R_n^N(t))/L} = L\left(\sum_{n=1}^{N}\frac{W_n^N(t)}{R_n^N(t)}(1 - K^N(t))\right)^{-1} \quad (2.11)$$

2.2 Reformulation in terms of a measure-valued process

Classes of connections: We will assume there are d classes of connections K_c, $c = 1, \ldots, d$ and all connections in class c have the same transmission time T_c. Hence $R_n^N = R_c^N$ for all $n \in K_c$. We will also assume the proportion of the N connections in class c is κ_c^N. In addition to Assumptions 1 we assume

Assumption 2 *Assumptions on connection classes:*

5) The proportion of users in the class c : $\kappa_c^N \to \kappa_c$ for $c = 1, \ldots, d$ as $N \to \infty$.

6) Let $\mu_c^N(0)$ be the histogram of windows of connections from class c at time 0. We suppose that for all c, $\mu_c^N(0)$ converges weakly to $\mu_c(0)$ as $N \to \infty$, where the support of $\mu_c(0)$ is concentrated on $[0, W_{max}]$.

Measure-valued process: The system (2.7), (2.8) can be solved pathwise from jump point to jump point. In order to study the limiting behavior of the system as the number of connections N goes to infinity, we will first define empirical process (see Dawson [4]) of those connections in class K_N^c. For any Borel set A define

$$M_c^N(t,A) \quad := \quad \frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\chi_A(W_n^N(t))\chi\{n\in K_c\} \tag{2.12}$$

to be the associated probability-measure-valued process taking values in $M_1(I\!R^+)$, the set of probability measures on $I\!R^+ = [0,\infty)$ furnished with the topology of weak convergence.

Reformulating (2.7): Let $\langle g,\mu\rangle = \int g(w)\mu(dw)$ and $\langle Id,\mu\rangle = \int w\mu(dw)$ so

$$m_c(s) := \langle Id, M_c^N(s)\rangle = \frac{1}{\kappa_c^N N}\sum_{n=1}^{N}W_n^N(s)\chi\{n\in K_c\}.$$

Then for each class,

$$\begin{aligned}&\langle g, M_c^N(t)\rangle - \langle g, M_c^N(0)\rangle\\ &= \frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\chi\{n\in K_c\}\int_0^t\left[\frac{dg_c}{dw}(W_n^N(s))\frac{1}{R_c^N(s)} + \left(g_c(W_n^N(s^-)/2) - g_c(W_n^N(s^-))\right)dN_n(\Lambda_n(s))\right]ds\end{aligned}$$

In Section 6, we consider the limit of the above as N goes to infinity to obtain an equation for the evolution for the evolution of the distribution of the windows.

Reformulating (2.8): For $Q^N(t) < q_{max}$, $K^N(t) = F(Q^N(s))$ and

$$\begin{aligned}&Q^N(t) - Q(0) \qquad (2.13)\\ &= \int_0^t\left[\sum_{c=1}^{d}\kappa_c^N\langle Id, M_c^N(s)\rangle\frac{(1-K^N(s))}{R_c^N(s)} - L\right.\\ &\quad - \left(\sum_{c=1}^{d}\kappa_c^N\langle Id, M_c^N(s)\rangle\frac{(1-K^N(s))}{R_c^N(s)} - L\right)^+\chi\{Q^N(t)=q_{max}\}\\ &\quad \left. + \left(\sum_{c=1}^{d}\kappa_c^N\langle Id, M_c^N(s)\rangle\frac{(1-K^N(s))}{R_c^N(s)} - L\right)^-\chi\{Q^N(s)=0\}\right]ds \qquad (2.14)\end{aligned}$$

where

$$R_c^N(t) = T_c + Q^N(t - R_c^N(t))/L, \tag{2.15}$$

If Q^N jitters at q_{max} then the loss rate $K^N(t)$ is given by

$$\sum_{c=1}^{d} \kappa_c^N \langle Id, M_c^N(t)\rangle \frac{(1-K^N(t))}{R_c^N(t)} = L$$

How the future is determined: The sequence $\{N_n(t); n = 1, \ldots\}$ of independent Poisson processes with intensity 1 was defined on a probability space $\{\Omega, \mathcal{F}, P\}$. $\mathbf{W}^N(t) \equiv (W_1^N(t), \ldots W_N^N(t))$, $Q^N(t)$ and $M^N(t) \equiv (M_1^N(t), \ldots, M_d^N(t))$ can be constructed path by path as processes defined on $\{\Omega, \mathcal{F}, P\}$ taking values in $(I\!R^+)^\infty$, $I\!R^+$ and $M_1(I\!R^+)^d$. It suffices to assume $W_n^N(t) = w_n(0)$ for $t \leq 0$ and build the solution up one round trip at a time.

3 Summary of the Mean-Field Limit

We wish to show $(\mathbf{W}^N(t), Q^N(t))$ converges as $N \to \infty$. In Section 4 we first prove the existence of the following limit.

Theorem 2 *If Assumptions [1]-[6] hold then there exists a unique strong solution $(\mathbf{W}, Q, (M_1, \ldots, M_d))$ to the following system. For $Q(t) < q_{max}$, $K(t) = F(Q(t))$ and*

$$\begin{aligned} &Q(t) - Q(0) \\ &= \int_0^t \left[\sum_{c=1}^{d} \kappa^c \langle Id, M_c(s)\rangle \frac{(1-K(s))}{R_c(s)} - L \right. \\ &\quad - \left(\sum_{c=1}^{d} \kappa^c \langle Id, M_c(s)\rangle \frac{(1-K(s))}{R_c(s)} - L \right)^+ \chi\{Q(s) = q_{max}\} \\ &\quad \left. + \left(\sum_{c=1}^{d} \kappa^c \langle Id, M_c(s)\rangle \frac{(1-K(s))}{R_c(s)} - L \right)^- \chi\{Q(s) = 0\} \right] ds. \end{aligned} \tag{3.16}$$

When $Q(t) = q_{max}$ then $K(t)$ satisfies

$$\sum_{c=1}^{d} \kappa^c \langle Id, M_c(s)\rangle \frac{(1-K(t))}{R_c(t)} = L$$

Each window evolves according to

$$dW_n(t) = \frac{1}{R_n(t)}dt - \frac{W_n(t^-)}{2}dN_n(\Lambda_n(t)), \tag{3.17}$$

where $W_n(0) = w_n(0)$, $n = 1, \ldots$ *are specified, where*

$$\Lambda_n(t) = \int_0^t \lambda_n(s)ds \text{ where } \lambda_n(s) = (1 - \dot{R}_n(s))\frac{W_n(s - R_n(s))}{R_n(s - R_n(s))}K(s - R_n(s))$$

and where $M_c(t)$ *is defined by*

$$\langle g, M_c(t)\rangle = \lim_{N\to\infty} \frac{1}{\kappa_c^N N}\sum_{n=1}^N g(\mathcal{W}_n^N(t))\chi\{n \in K_c\}.$$

so

$$\overline{W}_c(s) = \lim_{N\to\infty} \frac{1}{N\kappa_c^N}\sum_{n=1}^N W_n(s)\chi\{n \in K_c\}.$$

Consequently from (3.16)

$$\begin{aligned} &Q(t) - Q(0) \\ &= \int_0^t \Bigg[\sum_{c=1}^d \kappa_c \frac{\overline{W}_c(s)}{R_c(s)}(1 - K(s)) - L \\ &\quad - \left(\sum_{c=1}^d \kappa_c \frac{\overline{W}_c(s)}{R_c(s)}(1 - K(s)) - L\right)^+ \chi\{Q(s) = q_{max}\} \\ &\quad + \left(\sum_{c=1}^d \kappa^c \frac{\overline{W}_c(s)}{R_c(s)}(1 - K(s)) - L\right)^- \chi\{Q(s) = 0\}\Bigg] ds. \end{aligned} \tag{3.18}$$

The solutions $Q(t)$ *and* $\mathbf{R}(t) = (R_1(t), R_2(t), \ldots)$ *are deterministic as are the* $M_c(t)$. *Finally the components of* $\mathbf{W}$ *are independent processes.*

In Section 5 we will prove Theorem 3 and show that there is only one strong solution to (3.18), (3.17) and that in fact the solution to (2.7) and (2.8) converges to this strong solution.

Theorem 3 *If Assumptions [1]-[6] hold then* $M_c^N(t) \Rightarrow M_c(t)$, $Q^N(t) \to Q(t)$ *and* $K^N(t) \to K(t)$ *where* $M_c(t)$, $q(t)$ *and* $K(t)$ *are continuous deterministic functions of* $t \in I\!R^+$ *into* $M_1(I\!R^+)$, $I\!R^+$ *and* $I\!R^+$ *respectively given in Theorem 2.*

Finally in Section 6 we establish Theorem 1 which was the main conjecture in [1].

4 Existence of a limit

In this section we show the existence of the solution to (3.18), (3.17).

4.1 Modified system

First we introduce a modified system where Q^N is forced to be deterministic by modifying the equation for the evolution of the queue to (4.19). Then we extract a deterministic limit that turns to be a limit of our initial system.

$\mathcal{W}^N = (\mathcal{W}_1^N, \ldots, \mathcal{W}_N^N)$ satisfies the analogue of (2.7). For $\mathcal{Q} < q_{max}$, let $\mathcal{K}^N(t) = F(\mathcal{Q}^N(s))$ and

$$\begin{aligned}
&\mathcal{Q}^N(t) - Q(0) \qquad (4.19)\\
&= \int_0^t \Bigg[\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(s)\rangle)\frac{(1-\mathcal{K}^N(s))}{\mathcal{R}_c^N(s)} - L \\
&\quad - \left(\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(s)\rangle)\frac{(1-\mathcal{K}^N(s))}{\mathcal{R}_c^N(s)} - L\right)^{+} \chi\{\mathcal{Q}(s) = q_{max}\} \\
&\quad + \left(\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(s)\rangle)\frac{(1-\mathcal{K}^N(s))}{\mathcal{R}_c^N(s)} - L\right)^{-} \chi\{\mathcal{Q}(s) = 0\}\Bigg] ds
\end{aligned}$$

where we take E to mean $E(\cdot|\mathcal{F}_0)$, where $\mathcal{W}_n(0) = w_n(0)$, where

$$\mathcal{R}_c^N(t) = T_c + \mathcal{Q}^N(t - \mathcal{R}_c^N(t))/L, \tag{4.20}$$

and where

$$\langle Id, \mathcal{M}_c^N(s)\rangle = \frac{1}{\kappa_c^N N}\sum_{n=1}^{N} \mathcal{W}_n^N(s)\chi\{n \in K_c\}.$$

If $\mathcal{Q}^N = q_{max}$ then the loss rate $\mathcal{K}^N(t)$ is given by

$$\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(s)\rangle)\frac{(1-\mathcal{K}^N(t))}{R_c^N(t)} \quad = \quad L \tag{4.21}$$

Solution for a given N: Let $t_0^c = 0$, $t_{k+1}^c = t_k^c + T_c + \mathcal{Q}^N(t_k^c)/L$ such that $t_{k+1}^c - \mathcal{R}_c^N(t_{k+1}^c) = t_k^c$. As long as we can define it, the sequence $(t_k^c)_k$ is increasing. Define

$\Phi^c(t)$ = the first k such that $t_k^c > t$. We will construct our solution by recurrence from time t_i to t_{i+1} by defining $t_{i+1} = min_c(t^c_{\Phi^c(t_i)})$ starting from time $t_0 = 0$.

At time t_0, we suppose $\mathcal{W}^N(t)$ and $\mathcal{Q}^N(t)$ are given (perhaps constant) for $t \leq 0 = t_0$. We suppose $(\mathcal{W}^N(t), \mathcal{Q}^N(t)$ is defined for $t \leq t_i$ where t_i is a time such that $t_i = t_k^c$ for some c and some k. This is certainly true at time t_0. Then $\Phi^c(t_i)$ and $t^c_{\Phi^c(t_i)}$ are defined for all classes as is t_{i+1}.

Then if $t \leq t_{i+1}$,

$$\Lambda_n^N(t) = \int_0^t (1 - \dot{\mathcal{R}}_n^N(s)) \frac{\mathcal{W}_n^N(s - \mathcal{R}_n^N(s))}{\mathcal{R}_n^N(s - \mathcal{R}_n^N(s))} F(\mathcal{Q}^N(s - \mathcal{R}_n^N(s)))ds$$

can be defined, because for each class $s - \mathcal{R}_c^N(s) \leq t_i$ for $s \leq t_{i+1}$ by the definition of t_{i+1} (recall (2.11)). Hence the trajectories $\mathcal{W}^N(t)$ are defined on $[0, t_{i+1}]$, They are bounded and measurable thus the expectations can be defined and hence $\mathcal{Q}^N(t)$ can be defined. We have therefore check the induction hypothesis up to time t_{i+1}.

To conclude we need to show that $t_i \to \infty$. Notice that $t_{i+d+1} \geq \min\{t^c(\Phi^c(t_i) + 1) : c = 1, \ldots d\}$ because otherwise the $d+1$ values $t_{i+j}, j = 1, \ldots, d+1$ must be chosen among the d values $\{t^c(\Phi^c(t_i))\}$ and this is impossible. We conclude $t_{i+d+1} \geq t_i + T_{min}$ and therefore $t_i \to \infty$ as $i \to \infty$.

Lipschitz continuity of $E\langle Id, M_c^N(t)\rangle$ on $[0,T]$:

$$\begin{aligned}
&|E\langle Id, \mathcal{M}_c^N(t+h)\rangle - E\langle Id, \mathcal{M}_c^N(t)\rangle| \\
&\leq \frac{1}{\kappa_c^N N} \sum_{n=1}^N E|\mathcal{W}_n^N(t+h) - \mathcal{W}_n^N(t)|\chi\{n \in K_c\} \\
&\leq \frac{h}{T_{min}} \frac{1}{\kappa_c^N N} \sum_{n=1}^N P(\mathcal{W}_n^N(s^-) = \mathcal{W}_n^N(s), t \leq s \leq t+h)\chi\{n \in K_c\} \qquad (4.22) \\
&+ a(T) \frac{1}{\kappa_c^N N} \sum_{n=1}^N P(\mathcal{W}_n^N(s^-) \neq \mathcal{W}_n^N(s), \text{ for some } t \leq s \leq t+h)\chi\{n \in K_c\} \qquad (4.23)
\end{aligned}$$

The second term arises because even multiple jumps will create a difference less than the maximum window size.

(4.22) is less than h/T_{min} and tends to zero as $h \to 0$. (4.23) is bounded by the probability a window makes a jump in an interval of length h. The intensity function $\lambda^N(t)$ is bounded by $\overline{\lambda}(t) = a(t)/T_{min}$ so the probability of a jump in an interval of length h is bounded by $1 - \exp(a(T)h/T_{min})$. Hence, for some constant B, $E\langle Id, M_c^N(t)\rangle$ is $B-$Lipschitzian uniformly for $N \in \mathbb{N}$ and $t \in [0,T]$.

$E\langle Id, M_c^N(t)\rangle$ **is bounded away from zero:** Taking expectations of (2.7) gives

$$\begin{aligned}
&E\mathcal{W}_n^N(t) - w_n(0)\\
&= E\int_0^t \left[\frac{1}{\mathcal{R}_n^N(s)}ds - E\frac{\mathcal{W}_n^N(s^-)}{2}d\Lambda_n^N(s)\right]\\
&\geq \frac{t}{T_{min}+q_{max}/L} - E\int_0^t \left[\frac{\mathcal{W}_n^N(s^-)}{2}(1-\dot{\mathcal{R}}_n^N(s))\frac{\mathcal{W}_n^N(s-\mathcal{R}_n^N(s))}{\mathcal{R}_n^N(s-\mathcal{R}_n^N(s))}F(\mathcal{Q}^N(s-\mathcal{R}_n^N(s)))\right]ds\\
&\geq \frac{t}{T_{min}+q_{max}/L} - \int_0^t \frac{E\mathcal{W}_n^N(s^-)}{2}\frac{a(s)}{T_{min}}ds.
\end{aligned}$$

Hence the rate of increase of $E\mathcal{W}_n^N(t)$ is bounded below by

$$\frac{1}{T_{min}+q_{max}/L} - \frac{E\mathcal{W}_n^N(t^-)}{2}\frac{a(t)}{T_{min}}.$$

Since the above bound is positive when $E\mathcal{W}_n^N(t)$ is small the result follows.

4.2 Existence of a limit for the modified system

In this section we shall be extracting subsequences of sequences but we won't reflect this in our notation until the end of this section.

Extraction of a limit for $\mathcal{Q}_N$ and $E\langle Id, M_c^N(t)\rangle$: $\mathcal{Q}_N(t)$ is deterministic. Moreover, the integrand in (4.19) is bounded by a constant B because the window sizes up to time t are bounded by $a(T)$ and the RTT in greater than $T_{min} > 0$. Hence $\mathcal{Q}_N$ is $B-$Lipschitzian uniformly for $N \in \mathbb{N}$ and $t \in [0,T]$. It follows that there is a subsequence $N(\ell)$ and a $B-$Lipschitzian function $\mathcal{Q}(t)$ such that $\mathcal{Q}_{N(\ell)}(t) \to \mathcal{Q}(t)$ uniformly using the Ascoli-Arzelà Theorem plus the fact that a pointwise limit of Lipschitz functions is Lipschitz.

We showed above that $E\langle Id, M_c^N(t)$ is Lipschitz so again using the Ascoli-Arzelà Theorem we can take a further subsequence of $N(\ell)$ such that for all c, $E\langle Id, M_c^N(t)$ converges uniformly to a $B-$Lipschitz function $m_c(t)$.

Convergence of the RTT: As a direct consequence of the convergence of $\mathcal{Q}$, $\mathcal{R}_n^N(t)$ converges uniformly to $\mathcal{R}_n(t)$ where $\mathcal{R}_n(t) = T_n + \mathcal{Q}(t)/L$. The fact that the $\mathcal{Q}^N$s are $B-$Lipschitz implies the RTTs are $\frac{B}{L}-$Lipschitz and $\frac{1}{RTT}$s are $\frac{B}{LT_{min}^2}-$Lipschitz.

Convergence of the $\mathcal{K}^N(t)$: If $\mathcal{Q}(t) < q_{max}$ then $\mathcal{K}^N(t) = F(\mathcal{Q}^N(t))$ for N large enough and hence converges uniformly to $\mathcal{K}(t) = F(\mathcal{Q}(t))$. On the other hand if $\mathcal{Q}(t) = q_{max}$ then $\mathcal{K}^N(t)$ solves (4.21). By the above

$$\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(t)\rangle) \to \sum_{c=1}^{d} \kappa_c m_c(t).$$

Since $\sum_{c=1}^{d} \kappa_c^N (E\langle Id, \mathcal{M}_c^N(t)$ is bounded uniformly away from zero it follows that $\mathcal{K}^N(t)$ converges on an interval where $Q(t) = q_{max}$ to the Lipschitz function $\mathcal{K}(t)$ satisfying

$$\sum_{c=1}^{d} \kappa_c m_c(t) \frac{(1-\mathcal{K}(t))}{R_c(t)} \quad = \quad L$$

Pathwise convergence of $\mathcal{W}_n^N$ in the Skorohod metric: We fix some coordinate n. Clearly $\mathcal{W}_n^N(t) \leq a(T)$ for all $0 \leq t \leq T$ so it follows that $\lambda_n^N(t)$ is uniformly bounded by $\overline{\lambda}(t) = a(t)/T_{min}$ for all $0 \leq t \leq T$ and all $N \geq n$. Consequently $N_n(\Lambda_n^N(T)) \leq N_n(\overline{\Lambda}(T))$ where $\overline{\Lambda}(T) = \int_0^T \overline{\lambda}(t)dt$ so, if for some trajectory ω of N_n, $N_n(\overline{\Lambda}(T)) \leq m$ then $N_n(\Lambda_n^N(T)) \leq m$ for all N.

We now show that the sequence $\mathcal{W}_n^N(t)(\omega)$ indexed by N is a compact sequence in $D[0,T]$ with the Skorohod topology. Let U_n^k denote the k^{th} arrival time of $N_n(t)$. Let L_n^N denote the inverse of the function Λ_n^N. We first show that the points $\{L_n^N(U_n^k); k = 1, \ldots m\}$ are separated by some value Δ for N sufficiently large. Suppose on the contrary that there is a k and a subsequence such that $\lim_{N\to\infty}(L_n^N(U_n^{k+1}) - L_n^N(U_n^k)) = 0$. This would imply that

$$\begin{aligned} U_n^{k+1} - U_n^k &= \Lambda_n^N(L_n^N(U_n^{k+1}) - \Lambda_n^N(L_n^N(U_n^k) \\ &\leq \int_{L_n^N(U_n^k)}^{L_n^N(U_n^{k+1})} \overline{\lambda}(t)dt \to 0 \end{aligned}$$

as $N \to \infty$ and this is impossible.

We now apply Theorem 14.3 in [2]. Since there are at most some constant number m jumps of $\mathcal{W}^N$ for this sample path that are isolated by a distance Δ it suffices to consider a partition $\{t_i\}$ of $[0,T]$ with mesh size $\delta < \Delta$ which includes the jumps $L_n^N(U_n^k)$. Between jumps the oscillation of $\mathcal{W}_n^N$ is $w_x[t_{i-1}, t_i) < (t_i - t_{i-1})/T_{min}$ uniformly in N. Hence $w'_x(\delta)$ as defined in Section 14 in [2] tends to zero as $\delta \to 0$. This gives (14.33) in [2]. Condition (14.32) in [2] follows because $\mathcal{W}_n^N(t) \leq a(T)$

for all N. We conclude that $\mathcal{W}_n^N(t)(\omega)$ is a compact sequence in $D[0,T]$. Using the diagonalization method we can pick a subsequence $N(\ell) \equiv N$ such that for all n, $\mathcal{W}_n^N(t)(\omega)$ converges in the Skorohod topology on $D[0,T]$ to a limit $\mathcal{W}_n(t)(\omega) \in D[0,T]$.

Equation for the limit $\mathcal{W}_n$: Since $\mathcal{R}_n^N(s))$ and $\mathcal{K}^N(s-\mathcal{R}_n^N(s)))$ converge uniformly to $\mathcal{R}_n(s))$ and $\mathcal{K}(s-\mathcal{R}_n(s)))$ respectively it follows that $\Lambda_n^N(t)$ converges uniformly to

$$\Lambda_n(t) = \int_0^t (1-\dot{\mathcal{R}}_n(s))\frac{\mathcal{W}_n(s-\mathcal{R}_n(s))}{\mathcal{R}_n(s-\mathcal{R}_n(s))}\mathcal{K}(s-\mathcal{R}_n(s)))ds$$

on $[0,T]$ where

$$\mathcal{R}_n(t) = T_n + \mathcal{Q}(t-\mathcal{R}_n(t))/L. \tag{4.24}$$

Hence the point process $N_n(\Lambda_n^N(t))$ converges in the Skorohod topology on $D[0,T]$ to $N_n(\Lambda_n(t))$.

It therefore follows that both sides of

$$\mathcal{W}_n^N(t) - w_n(0) = \int_0^t \left[\frac{1}{\mathcal{R}_n^N(s)}ds - \frac{\mathcal{W}_n^N(s^-)}{2}dN_n(\Lambda_n^N(s))\right]$$

converge in the Skorohod topology on $D[0,T]$ to give

$$\mathcal{W}_n(t) - w_n(0) \quad = \quad \int_0^t \left[\frac{1}{\mathcal{R}_n(s)}ds - \frac{\mathcal{W}_n(s^-)}{2}dN_n(\Lambda_n(s))\right]. \tag{4.25}$$

4.3 Equations for the limit $(\mathcal{W},\mathcal{Q})$:

Equation for $\mathcal{Q}$: Since $\mathcal{Q}(t)$ is deterministic the equations in (4.25) are independent. This in turn means that

$$E\overline{\mathcal{W}}_c(s) = \overline{\mathcal{W}}_c(s) = \lim_{N\to\infty}\frac{1}{\kappa_c^N N}\sum_{n=1}^N \mathcal{W}_n(s)\chi\{n\in K_c\}.$$

This essentially follows from the law of large numbers. It suffices to consider the $\mathcal{W}_n(s) = f(\mathcal{W}_n(0),N_n)$ defined by (4.25). f is defined on the space $([0,\infty)\times D([0,T]))$ where $0\le s\le T$. with metric $m = e\oplus d$ where e is the Euclidean metric on $[0,\infty)$ and d is the Skorohod metric on $D([0,T])$. Since the probability of a jump precisely at time s is zero the set of discontinuities of the function f on $([0,\infty)\times D([0,T]))$ relative to the metric m has probability zero. By hypothesis

$$\lim_{N\to\infty}\frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\delta_{\mathcal{W}_n(0)}\chi\{n\in K_c\}\to\mu_c$$

and the N_n are i.i.d independent Poisson processes so the empirical measure of the pairs $(\mathcal{W}_n(0),N_n))$ converges; i.e.

$$\frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\delta_{(\mathcal{W}_n(0),N_n)}\chi\{n\in K_c^N\}\to\mu_c\otimes\nu$$

where ν is the distribution of N_n in $D([0,T])$. The result now follows since f is bounded and the set of discontinuities has probability zero relative to the limiting measure.

In addition the above means that $\mathcal{M}_c^N(t)$ converges weakly to $\mathcal{M}_c(t)$ almost surely P; for any continuous function with compact support define the limiting measure $\mathcal{M}_c(t)$ by

$$\langle g,\mathcal{M}_c(t)\rangle=\lim_{N\to\infty}\frac{1}{\kappa_c^N N}\sum_{n=1}^{N_\ell}g(\mathcal{W}_n^N(t))\chi\{n\in K_c\}.$$

The deterministic limit exists almost surely by the argument above. This also means that $m_c(t)=\langle Id,M_c(t)\rangle$ so when $\mathcal{Q}(t)=q_{max}$, $K(t)$ satisfies

$$\sum_{c=1}^{d}\kappa_c^N\langle Id,\mathcal{M}_c(s)\rangle\frac{(1-\mathcal{K}(t))}{R_c(t)} \quad = \quad L.$$

Moreover,

$$\begin{aligned}\overline{\mathcal{W}}_c(t) \quad &:= \quad \lim_{N(\ell)\to\infty}\frac{1}{\kappa_c^{N(\ell)}N(\ell)}\sum_{n=1}^{N(\ell)}\mathcal{W}_n^{N(\ell)}(t)\chi\{n\in K_c\}\\ &= \quad \langle Id,\mathcal{M}_c(t)\rangle.\end{aligned}$$

Hence along the subsequence $N(\ell)$ we obtain an almost sure limit point $(\mathcal{W}, \mathcal{Q}, (\mathcal{M}_1, \dots \mathcal{M}_d))$ which satisfies the modified system: (4.25) and

$$\begin{aligned} &\mathcal{Q}(t) - \mathcal{Q}(0) && (4.26) \\ = &\int_0^t \Bigg[\sum_{c=1}^{d} \kappa_c \frac{\overline{\mathcal{W}}_c(s)}{\mathcal{R}_c(s)} (1 - \mathcal{K}(s)) - L \\ &- \left(\sum_{c=1}^{d} \kappa_c \frac{\overline{\mathcal{W}}_c(s)}{\mathcal{R}_c(s)} (1 - \mathcal{K}(t)) - L \right)^+ \chi\{\mathcal{Q}(t) = q_{max}\} \\ &+ \left(\sum_{c=1}^{d} \kappa_c \frac{\overline{\mathcal{W}}_c(s)}{\mathcal{R}_c(s)} (1 - F(\mathcal{Q}(s))) - L \right)^- \chi\{\mathcal{Q}(s) = 0\} \Bigg] ds \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} \overline{\mathcal{W}}_c(t) &:= \lim_{N \to \infty} \frac{1}{\kappa_c^N N} \sum_{n=1}^{N_\ell} \mathcal{W}_n^N(t) \chi\{n \in K_c\} \\ &= \langle Id, \mathcal{M}_c(t) \rangle. \end{aligned}$$

We call the above a strong solution associated with the subsequence N_ℓ. Hence the solution to (4.26) and (4.25) proves the existence of a strong solution of the system in Theorem 2.

Extension to the timeout and slow-start phases: If we consider the extended system with timeouts and slow-start we have to define $N_c^A(t)$, the proportion of the connections from class c in congestion avoidance at time t. There will be similar proportions $N_c^U(t)$ in timeout and $N_c^S(t)$ in slow-start. The equation for the queue (neglecting boundary terms) becomes

$$\frac{dQ^N(t)}{dt} = \sum_{c=1}^{d} [\kappa_c^N N_c^A(t) \langle Id, M_c^N(t) \rangle + \kappa_c^N N_c^S(t) \langle Id, S_c^N(t) \rangle] - L$$

where $M_c^N(t)$ is the histogram of the window sizes of connections in congestion avoidance and $S_c^N(t)$ is the histogram of the window sizes of connections in slow start.

We can force the queue to be deterministic by considering the modified system (again neglecting boundary terms):

$$\frac{d\mathcal{Q}^N(t)}{dt} = \sum_{c=1}^{d}[\kappa_c^N E(N_c^A(t))E\langle Id, M_c^N(t)\rangle + \kappa_c^N E(N_c^S(t))E\langle Id, S_c^N(t)\rangle] - L$$

The window equations for the modified system are uncoupled as before. We can again pick subsequences so that $\mathcal{Q}^N(t)$ converges and then further subsequences so that $E(N_c^A(t))$, $E(N_c^U(t))$ and $E(N_c^S(t))$ converge. As before the limiting system is in fact a strong solution to the extended system.

5 Uniqueness of strong solutions

We have constructed a strong solution $(\mathbf{W}, \mathbf{R}, Q, M)$ to (3.18) and (3.17). We first prove that the system $\mathbf{W}^N, Q^N, M^N$ converges to this strong solution up to the stopping time when Q first reaches 0 or q_{max}. We therefore start with the assumption:

- $Q_N(0) = q(0)$ where $q_{min} < q(0) < q_{max}$.

Define ρ to the (deterministic) time when $Q(t)$ first hits 0 or q_{max} and define ρ^N to the stopping time when $Q^N(t)$ first hits 0 or q_{max}. Define the distance between the the marginal process $\mathbf{W}^N(t) \equiv (W_1^N(t), \ldots W_N^N(t))$, $Q^N(t)$ and $M^N(t) \equiv (M_1^N(t), \ldots, M_d^N(t)))$. and the limit processes up to the stopping time $\rho \wedge \rho^N$. For any $t \leq \rho$ define

$$||\mathbf{W}^N(t \wedge \rho^N) - \mathbf{W}(t \wedge \rho^N)|| = \frac{1}{N}\sum_{n=1}^{N} E \sup_{0 \leq \tau \leq t \wedge \rho^N} |W_n^N(\tau) - W_n(\tau)|$$

where τ is a stopping time with respect to $\mathcal{F}_t$. Define

$$D_N(t) \equiv E \sup_{\tau \leq t \wedge \rho^N} |Q^N(\tau) - Q(\tau)| + ||\mathbf{W}^N(t) - \mathbf{W}(t)||.$$

We will establish a Gronwall inequality: $D_N(t) \leq B_N + C\int_0^t D_N(s)ds$ for $t \leq \rho$ where $B_N < \epsilon$ for N sufficently large where ϵ is arbitrarily small and where C will be a canonical constant throughout this calculation. It will then immediately follows that $\lim_{N\to\infty} D_N(t) = 0$ on $[0, T]$.

This will mean that $Q^N(t)$ will converge in probability to $Q^N(t)$ in the uniform norm and that ρ^N converges to ρ in probability. Moreover, $M_c^N(t)$ converges weakly to $M_c(t)$ for any $t \leq \rho$ since for any Lipschitz function g

$$\begin{aligned}
&\limsup_{N\to\infty} |E\langle g, M_c^N(t)\rangle - E\langle g, M_c(t)\rangle| \\
&= \limsup_{N\to\infty} |E\left[\frac{1}{\kappa_c^N N}\sum_{n=1}^N g(W_n^N(t))\chi\{n \in K^c\} - \lim_{N\to\infty}\frac{1}{\kappa_c^N N}\sum_{n=1}^N g(W_n(t))\chi\{n\in K_c\}\right]| \\
&\leq \limsup_{N\to\infty} |\frac{1}{\kappa_c^N N}\sum_{n=1}^N E\left[g(W_n^N(t)) - g(W_n(t))\right]\chi\{n\in K_c\}| \\
&\leq \limsup_{N\to\infty} \frac{1}{\kappa_c^N N}\sum_{n=1}^N E|g(W_n^N(t)) - g(W_n(t))| \\
&\leq C_g \limsup_{N\to\infty}\frac{1}{N}\sum_{n=1}^N E|W_n^N(t) - W_n(t)| \\
&\qquad \text{where } C_g \text{ is the Lipschitz constant divided by } \min\{\kappa_N^c\}, \\
&\leq C_g||\mathbf{W}^N(t) - \mathbf{W}(t)|| \to 0.
\end{aligned}$$

Since the equation (3.18) only depends on $\langle Id, M_c(s)\rangle, c = 1,\ldots,d$ it will follow that $Q^N(t)$ converges uniformly to $Q(t)$ almost surely. Moreover $R_c^N(t)$ will converge uniformly to $R_c(t)$. Since the window equations (3.17) depend on Q and R_c it will follow that for any n, $W_n^N(t)$ converges uniformly to $W_n(t)$ almost surely.

Lemma 1 *The sample paths of both $Q^N(t)$ and $Q(t)$ form a subset of $C([0,T], I\!R^+)$ which has compact closure.*

Proof

The modulus of continuity of a sample path satisfies

$$\begin{aligned}
w(\delta) &= \sup_{|s-t|<\delta; 0\leq s<t\leq T} |Q(t) - Q(s)| \\
&\leq \sup_{|t-s|<\delta; s,t\leq T}\int_s^t |\sum_{c=1}^d \kappa^c \frac{\overline{W}_c(s)}{R_c(s)}(1 - F(Q(s)) - L|ds \\
&\leq \delta sup_{0\leq s\leq T}(\frac{a(s)}{T_{min}} + L) \\
&\leq C\delta
\end{aligned}$$

where C denotes a constant. Since $Q(0) = q(0)$ the result follows from the Arzelà-Ascoli Theorem. The same proof works for Q^N.

■

Lemma 2 *For $0 \leq s \leq t \leq T$, we have*

$$\sup_{0\leq s\leq t} |R_n^N(s) - R_n(s)|, \sup_{0\leq s\leq t} |R_n^N(s - R_n^N(s)) - R_n(s - R_n(s))|,$$
$$\sup_{0\leq s\leq t} |Q^N(s - R_n^N(s)) - Q(s - R_n(s))| \text{ and } \sup_{0\leq s\leq t} |F(Q^N(s - R_n^N(s))) - F(Q(s - R_n(s)))|$$

are bounded by $Csup_{0\leq s\leq t}|Q^N(s) - Q(s)|$.

Proof

The $R_n^N(s)$, $R_n^N(s - R_n^N(s))$, $Q^N(s - R_n^N(s))$ and $F(Q^N(s - R_n^N(s)))$ are continuous functions of the sample path of Q^N up to time s while $R_n(s)$, $R_n(s - R_n(s))$ $Q(s - R_n(s))$ and $F(Q(s - R_n(s)))$ are the same continuous functions of the sample path of Q up to time s. Hence by Lemma 1 there exists a constant C such that the inequality in the lemma holds. Moreover the same constant works for all $s \leq t$ and all N.

■

5.1 Convergence away from the boundaries

Define

$$\overline{\mathbf{S}}_N^N(s) = \frac{1}{N} \sum_{n=1}^{N} \frac{W_n^N(s)}{R_n^N(s)} = \sum_{c=1}^{d} \kappa_c^N \frac{\overline{W}_c^N(s)}{R_c^N(s)}.$$

where $\overline{W}_c^N(s)$ is the average window size of connections in K_c among the first $W_1^N, \ldots W_N^N$. Define $\overline{\mathbf{S}}_N(s)$ analogously from $\mathbf{W}$. Define $\overline{\mathbf{S}}(s) = \lim_{N\to\infty} \overline{\mathbf{S}}_N(s)$.

$$\begin{aligned}
& E \sup_{0\leq\tau\leq t\wedge\rho^N} |Q^N(\tau) - Q(\tau)| \\
\leq \; & E \sup_{0\leq\tau\leq t\wedge\rho^N} |\int_0^\tau \overline{\mathbf{S}}_N^N(s)(1 - F(Q^N(s)) - \overline{\mathbf{S}}(s)(1 - F(Q(s))ds| \\
\leq \; & E \sup_{0\leq\tau\leq t\wedge\rho^N} |\int_0^\tau \left(\overline{\mathbf{S}}_N^N(s) - \overline{\mathbf{S}}_N(s)\right)(1 - F(Q^N(s))ds| \\
& +E \sup_{0\leq\tau\leq t\wedge\rho^N} |\int_0^\tau \left(\overline{\mathbf{S}}_N(s) - \overline{\mathbf{S}}(s)\right)(1 - F(Q^N(s))ds| \\
& +E \sup_{0\leq\tau\leq t\wedge\rho^N} |\int_0^\tau \overline{\mathbf{S}}(s)(F(Q^N(s)) - F(Q(s))ds|
\end{aligned}$$

Hence,

$$
\begin{aligned}
E \sup_{0\le\tau\le t\wedge\rho^N} |Q^N(\tau)-Q(\tau)| \quad &\le \quad E \sup_{0\le\tau\le t\wedge\rho^N} \int_0^\tau \left(|\frac{1}{N}\sum_{n=1}^N [\frac{W_n^N(s)}{R_n^N(s)} - \frac{W_n(s)}{R_n(s)}]| \right) ds \\
&\quad + E \sup_{0\le\tau\le t\wedge\rho^N} \int_0^\tau |\overline{\mathbf{S}}_N(s) - \overline{\mathbf{S}}(s)| ds \\
&\quad + E \sup_{0\le\tau\le t\wedge\rho^N} \int_0^\tau \overline{\mathbf{S}}(s) |F(Q^N(s)) - F(Q(s))| ds \\
&\le \quad \int_0^t \frac{1}{N}\sum_{n=1}^N E\left(\sup_{0\le\tau\le s\wedge\rho^N} |\frac{W_n^N(s)}{R_n^N(s)} - \frac{W_n(s)}{R_n(s)}| \right) ds \\
&\quad + B_N(1) + \int_0^t E\left(\sup_{0\le\tau\le s\wedge\rho^N} \overline{\mathbf{S}}(s) |F(Q^N(s)) - F(Q(s))| \right) ds
\end{aligned}
$$

where $B_N(1) = E\int_0^t |\overline{\mathbf{S}}_N(s) - \overline{\mathbf{S}}(s)| ds$.

Next,

$$
\begin{aligned}
& E\left(\sup_{0\le\tau\le s\wedge\rho^N} |\frac{W_n^N(s)}{R_n^N(s)} - \frac{W_n(s)}{R_n(s)}| \right) \qquad (5.28) \\
\le\ & E\left(\sup_{0\le\tau\le s\wedge\rho^N} |W_n^N(s) - W_n(s)| \frac{1}{R_n^N(s)}] \right) + E[\left(\sup_{0\le\tau\le s\wedge\rho^N} |W_n(s)| \frac{1}{R_n^N(s)} - \frac{1}{R_n(s)}| \right) \\
\le\ & \frac{1}{T_{min}} E\left(\sup_{0\le\tau\le s\wedge\rho^N} |W_n^N(s) - W_n(s)| \right) + \frac{a(s)}{T_{min}^2} E\left(\sup_{0\le\tau\le s\wedge\rho^N} |R_n^N(s) - R_n(s)| \right) \\
\le\ & \frac{1}{T_{min}} E\left(\sup_{0\le\tau\le s\wedge\rho^N} |W_n^N(\tau) - W_n(\tau)| \right) + \frac{1}{T_{min}^2} a(s) C E\left(\sup_{0\le\tau\le s\wedge\rho^N} |Q^N(\tau) - Q(\tau)| \right)
\end{aligned}
$$

using Lemma 2.

Moreover,

$$
E\left(\sup_{0\le\tau\le s\wedge\rho^N} \overline{\mathbf{S}}(s) |F(Q^N(s)) - F(Q(s))| \right) \quad \le \quad \frac{a(s)}{T_{min}} L_F E\left(\sup_{0\le\tau\le s\wedge\rho^N} |Q^N(\tau) - Q(\tau)| \right)
$$

where L_F is the Lipschitz constant of F.

Hence,

$$
\begin{aligned}
& E\left(\sup_{0\le\tau\le t\wedge\rho^N}|Q^N(\tau)-Q(\tau)|\right) \\
\le\ & \int_0^t \frac{1}{T_{min}}\frac{1}{N}\sum_{n=1}^N E\left(\sup_{0\le\tau\le s\wedge\rho^N}|W_n^N(\tau)-W_n(\tau)|\right)ds \\
& +B_N+\int_0^t \frac{1}{T_{min}^2}a(s)CE\left(\sup_{0\le\tau\le s\wedge\rho^N}|Q^N(\tau)-Q(\tau)|\right)ds \\
& +\int_0^t \frac{1}{T_{min}}L_F a(s)E\left(\sup_{0\le\tau\le s\wedge\rho^N}|Q^N(\tau)-Q(\tau)|\right)ds \\
\le\ & B_N+C\int_0^t D_N(s)ds
\end{aligned}
$$

From (3.17),

$$
\begin{aligned}
& \frac{1}{N}\sum_{n=1}^N E\left(\sup_{0\le\tau\le t\wedge\rho^N}|W_n^N(\tau)-W_n(\tau)|\right) \\
\le\ & E\left(\sup_{0\le\tau\le t\wedge\rho^N}\int_0^\tau \frac{1}{N}\sum_{n=1}^N E|\frac{1}{R_n^N(s)}-\frac{1}{R_n(s)}|ds\right) \\
& +\frac{1}{2}\frac{1}{N}\sum_{n=1}^N E\left(\sup_{0\le\tau\le t\wedge\rho^N}|\int_0^\tau\left[W_n^N(s^-)dN_n(\Lambda_n^N(s))-W_n(s^-)dN_n(\Lambda_n(s))\right]|\right)
\end{aligned}
$$

However

$$
\int_0^\tau \frac{W_n^N(s^-)}{2}dN_n(\Lambda_n^N(s))=\int_{s=0}^\tau\int_{u=0}^\infty \frac{W_n^N(s^-)}{2}\chi_{[0,\lambda_n^N(s))}(u)N_n(du,ds)
$$

where N_n is a sequence of iid Poisson processes on $[0,\infty)^2$ with intensity one and $\lambda_n^N(s)$ is the derivative of $\Lambda_n^N(s)$. Consequently

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\le\tau\le t\wedge\rho^N} |\int_0^\tau \left[W_n^N(s^-)dN_n(\Lambda_n^N(s)) - W_n(s^-)dN_n(\Lambda_n(s))\right]|\right)\\
&\le \frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\le\tau\le t\wedge\rho^N}\int_0^\tau\int_{u=0}^{\infty} |W_n^N(s^-)\chi_{[0,\lambda_n^N(s))}(u) - W_n(s^-)\chi_{[0,\lambda_n(s))}(u)|N_n(du,ds)\right)\\
&= \frac{1}{N}\sum_{n=1}^{N} E\left(\int_0^{t\wedge\rho^N}\int_{u=0}^{\infty} |W_n^N(s^-)\chi_{[0,\lambda_n^N(s))}(u) - W_n(s^-)\chi_{[0,\lambda_n(s))}(u)|N_n(du,ds)\right)\\
&= \frac{1}{N}\sum_{n=1}^{N} E\left(\int_0^{t\wedge\rho^N}\int_{u=0}^{\infty} |W_n^N(s^-)\chi_{[0,\lambda_n^N(s))}(u) - W_n(s^-)\chi_{[0,\lambda_n(s))}(u)|duds\right)\\
&\le \frac{1}{N}\sum_{n=1}^{N} E\left(\int_0^{t\wedge\rho^N} |W_n^N(s^-) - W_n(s^-)|\lambda_n^N(s)\wedge\lambda_n(s)ds\right)\\
&\quad +\frac{1}{N}\sum_{n=1}^{N} E\left(\int_0^{t\wedge\rho^N} |W_n^N(s^-)\vee W_n(s^-)|\cdot|\lambda_n^N(s) - \lambda_n(s))|ds\right)\\
&\le \int_0^t \frac{a(s)}{T_{min}}\left[\frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\le\tau\le s\wedge\rho^N}|W_n^N(\tau) - W_n(\tau)|\right)\right]ds \qquad (5.29)\\
&\quad +\frac{1}{N}\sum_{n=1}^{N} E\left(\int_0^{t\wedge\rho^N} a(s)|\lambda_n^N(s) - \lambda_n(s))|ds\right) \qquad (5.30)
\end{aligned}$$

where $\lambda_n^N(s)$ and $\lambda_n(s)$ are less than $(w_n(0)+s/T_{min})/T_{min} = a(s)/T_{min}$.

Also

$$
\begin{aligned}
&|\lambda_n^N(s) - \lambda_n(s))| \\
&\leq \; |W_n^N(s - R_n^N(s)) - W_n(s - R_n^N(s))| \frac{1 - \dot{R}_n^N(s)}{R_n^N(s - R_n^N(s))} F(Q^N(s - R_n^N(s))) \\
&\quad + |W_n(s - R_n^N(s)) - W_n(s - R_n(s))| \frac{1 - \dot{R}_n^N(s)}{R_n^N(s - R_n^N(s))} F(Q^N(s - R_n^N(s))) \\
&+ \; |W_n(s - R_n(s))| \frac{1}{R_n^N(s - R_n^N(s))} - \frac{1}{R_n(s - R_n(s))} |(1 - \dot{R}_n^N(s)) F(Q^N(s - R_n^N(s))) \\
&+ \; W_n(s - R_n(s)) \frac{1 - \dot{R}_n^N(s)}{R_n(s - R_n(s))} |F(Q^N(s - R_n^N(s))) - F(Q(s - R_n(s)))| \\
&+ \; W_n(s - R_n(s)) \frac{1}{R_n(s - R_n(s))} F(Q(s - R_n(s)) |\dot{R}_n^N(s) - \dot{R}_n(s)|
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\lambda_n^N(s) - \lambda_n(s))| \;\leq\; & |W_n^N(s - R_n^N(s)) - W_n(s - R_n^N(s))|/T_n && (5.31)\\
& + |W_n(s - R_n^N(s)) - W_n(s - R_n(s))|/T_n && (5.32)\\
& + a(s)|R_n^N(s - R_n^N(s)) - R_n(s - R_n(s))|/T_n^2 && (5.33)\\
& + a(s)|F(Q^N(s - R_n^N(s))) - F(Q(s - R_n(s)))|/T_n && (5.34)\\
& + \frac{a(s)}{T_n} |\dot{R}_n^N(s) - \dot{R}_n(s)| && (5.35)
\end{aligned}
$$

We must bound $E\left(\int_0^{t\wedge\rho^N} |\lambda_n^N(s) - \lambda_n(s)| ds\right)$ so we must bound the expectation of the integral of each of the above terms. The first (5.31) satisfies

$$
\begin{aligned}
& E\left(\int_0^{t\wedge\rho^N} |W_n^N(s - R_n^N(s)) - W_n(s - R_n^N(s))| ds / T_n\right) \\
& \leq \; \frac{1}{T_{min}} \int_0^t E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} |W_n^N(\tau) - W_n(\tau)| ds\right)
\end{aligned}
$$

The second term (5.32) is bounded by

$$\begin{aligned}
&E\left(\int_0^{t\wedge\rho^N} |W_n(s-R_n^N(s)) - W_n(s-R_n(s))|ds\right)/T_n \\
\leq\ & E\left(\int_0^{t\wedge\rho^N}\int_{[s-R_n^N(s)\wedge s-R_n(s), s-R_n^N(s)\vee s-R_n(s)]} \frac{1}{R_n(u)} duds\right) \\
+\ & \frac{1}{2}E\left(\int_0^{t\wedge\rho^N}\int_{[(s-R_n^N(s))\wedge(s-R_n(s)),(s-R_n^N(s))\vee(s-R_n(s))]} W_n(u^-)dN_n(\Lambda_n(u))ds\right)
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^{t\wedge\rho^N} \chi\{(s-R_n^N(s))\wedge(s-R_n(s)) \leq u \leq (s-R_n^N(s))\vee(s-R_n(s))\}ds \\
&\quad = (u+\phi_n^N(u)\vee\phi_n(u))\wedge(t\wedge\rho^N) - (u+\phi_n^N(u)\wedge\phi_n(u))\wedge(t\wedge\rho^N).
\end{aligned}$$

where $\phi_n(u) = T_n + Q(u)/L$ and $\phi_n^N(u) = T_n + Q^N(u)/L$

Hence,

$$\begin{aligned}
&E\left(\int_0^{t\wedge\rho^N} |W_n(s-R_n^N(s)) - W_n(s-R_n(s))|ds\right)/T_n \\
\leq\ & \frac{1}{T_{min}}E\left(\int_0^{t\wedge\rho^N} |R_n(s)-R_n^N(s)|ds\right) \\
&+\frac{1}{2}E\left(\int_0^{t\wedge\rho^N} |\phi_n^N(u)\vee\phi_n(u) - \phi_n^N(u)\wedge\phi_n(u)|W_n(u^-)\lambda_n(u))du\right) \\
\leq\ & \frac{1}{T_{min}}E\left(\int_0^{t\wedge\rho^N} |R_n(s)-R_n^N(s)|ds\right) + \frac{1}{2}E\left(\int_0^{t\wedge\rho^N} |\phi_n^N(u)-\phi_n(u)|\frac{a^2(u)}{T_{min}}du\right) \\
\leq\ & \frac{C}{T_{min}}\int_0^t E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} |Q(s)-Q^N(s)|\right)ds + + \int_0^t \frac{1}{2}\frac{a^2(s)}{T_{min}L}E\left(\sup_{0\leq\tau\leq s\wedge\rho^N}|Q^N(\tau)-Q(\tau)|\right)ds \\
\leq\ & C\int_0^t E\left(\sup_{0\leq\tau\leq s\wedge\rho^N}|Q^N(\tau)-Q(\tau)|\right)ds
\end{aligned}$$

where C is a constant.

To bound the third term (5.33)

$$|R_n^N(s - R_n^N(s)) - R_n(s - R_n(s))| \leq C \sup_{\tau \leq s} |Q^N(\tau) - Q(\tau)|$$

by Lemma 2. Taking expectations gives

$$E\left(\int_0^{t\wedge\rho^N} \frac{a_n(s)}{T_n^2} |R_n^N(s - R_n^N(s)) - R_n(s - R_n(s))|\right) ds \leq C \int_0^t E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} |Q^N(\tau) - Q(\tau)|\right).$$

Similarly, to bound the fourth term (5.34)

$$\begin{aligned}
&\frac{a(s)}{T_n}|F(Q^N(s - R_n^N(s))) - F(Q(s - R_n(s)))| \\
&\leq C(|F(Q^N(s - R_n^N(s))) - F(Q(s - R_n^N(s)))| + |F(Q(s - R_n^N(s))) - F(Q(s - R_n(s)))|) \\
&\leq C \sup_{\tau\leq s} |Q^N(\tau) - Q(\tau)| + C|R_n^N(s) - R_n(s)| \\
&\leq C \sup_{\tau\leq s} |Q^N(\tau) - Q(\tau)|
\end{aligned}$$

by Lemma 2. Taking expectations shows the fourth term is bounded by

$$C \int_0^t E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} |Q^N(\tau) - Q(\tau)|\right).$$

Finally, to bound the fifth term (5.35) recall that from (2.11)

$$\begin{aligned}
&|\dot{R}_n^N(s) - \dot{R}_n(s)| \\
&= L\left(\sum_{n=1}^N \frac{W_n^N(s)}{R_n^N(s)}(1 - F(Q^N(s)))\right)^{-1} - L\left(\sum_{n=1}^N \frac{W_n(s)}{R_n(s)}(1 - F(Q(s)))\right)^{-1} \\
&\leq \frac{L}{(1-p_{max})^2 \overline{\mathbf{S}}_N^N(s)\cdot\overline{\mathbf{S}}_N(s)} \left|\sum_{n=1}^N \frac{W_n^N(s)}{R_n^N(s)}(1 - F(Q^N(s))) - \sum_{n=1}^N \frac{W_n(s)}{R_n(s)}(1 - F(Q(s)))\right|
\end{aligned}$$

Define

$$f_N = \chi\{\overline{\mathbf{S}}_N^N(s) < \delta \text{ or } \overline{\mathbf{S}}_N(s) < \delta \text{ for some } 0 \leq s \leq T.\}$$

Hence,

$$\begin{aligned}
&|\dot{R}_n^N(s) - \dot{R}_n(s)| \\
&\leq \frac{L}{(1-p_{max})^2\delta^2} \left|\sum_{n=1}^N \frac{W_n^N(s)}{R_n^N(s)}(1 - F(Q^N(s))) - \sum_{n=1}^N \frac{W_n(s)}{R_n(s)}(1 - F(Q(s)))\right| \\
&\quad + f_N \frac{L}{(1-p_{max})^2 \overline{\mathbf{S}}_N^N(s)\cdot\overline{\mathbf{S}}_N(s)} 2N\frac{a(s)}{T_{min}}.
\end{aligned}$$

As above,

$$
\begin{aligned}
&|\frac{W_n^N(s)}{R_n^N(s)}(1-F(Q^N(s)))-\frac{W_n(s)}{R_n(s)}(1-F(Q(s)))| \\
&\leq \; C\sup_{\tau\leq s}|W_n^N(\tau)-W_n(\tau)|+C\sup_{\tau\leq s}|Q^N(\tau)-Q(\tau)|.
\end{aligned}
$$

Taking expectations shows the fifth term is bounded by

$$
B_N(2)+CE\left(\int_0^{t\wedge\rho^N}[\sup_{\tau\leq s}|W_n^N(\tau)-W_n(\tau)|+\sup_{\tau\leq s}|Q^N(\tau)-Q(\tau)|]\right)ds
$$

where

$$
\begin{aligned}
B_N(2) &= 2E\left(f_N\left(\int_0^{t\wedge\rho^N}\left(\frac{a(s)}{T_n}\right)^2\frac{L}{(1-p_{max})^2\overline{\mathbf{S}}_N^N(s)\cdot\overline{\mathbf{S}}_N(s)}ds\right)\right) \\
&\leq CE\left(f_N\left(\int_0^T\frac{1}{\mathbf{S}_N^N(s)\cdot\overline{\mathbf{S}}_N(s)}ds\right)\right).
\end{aligned}
$$

Losses for connection n are generated at a maximum rate $a(T)$ on the interval $[0,T]$ so if $N_n(a(T)t)$ is stochastically larger than $N_n(\Lambda_n^N(t)$ for all $t<T$. Let E_N denote the indices n for $1\leq n\leq N$ such that $N_n(a(T)T)=0$. Both $\overline{\mathbf{S}}_N(s)$ and $\mathbf{S}_N^N(s)$ are bounded below by

$$
\frac{1}{T_{max}}\frac{1}{N}\sum_{n\in E_N}w_n(0).
$$

Also,

$$
f_N\leq\chi\{\frac{1}{T_{max}}\frac{1}{N}\sum_{n\in E_N}w_n(0)<\delta\}.
$$

Hence,

$$
\begin{aligned}
B_N(2) &\leq CTE\left(\{\frac{1}{N}\sum_{n\in E_N} w_n(0) < T_{max}\delta\}\left(\frac{1}{N}\sum_{n=1}^{N} w_n(0)\{n\in E_N\}\right)^{-2}\right) \\
&\leq CTE\left(\{\frac{1}{N}\sum_{n\in E_N} w_n(0) < T_{max}\delta\}\frac{1}{N}\sum_{n=1}^{N} w_n(0)^{-2}\{n\in E_N\}\right) \\
&\leq CTP(\frac{1}{N}\sum_{n\in E_N} w_n(0) < T_{max}\delta)^{1/2}(E(\left(\frac{1}{N}\sum_{n=1}^{N} w_n(0)^{-2}\{n\in E_N\}\right)^2)^{1/2} \\
&\leq CTP(\frac{1}{N}\sum_{n\in E_N} w_n(0) < T_{max}\delta)^{1/2}(E(\left(\frac{1}{N}\sum_{n=1}^{N} w_n(0)^{-4}\{n\in E_N\}\right))^{1/2} \\
&\leq CTP(\frac{1}{N}\sum_{n\in E_N} w_n(0) < T_{max}\delta)^{1/2}\left(\frac{p}{N}\sum_{n=1}^{N} w_n(0)^{-4}\right))^{1/2}
\end{aligned}
$$

where $p = \exp(-a(T)T)$. Next, by Assumption 2), $\frac{1}{N}\sum_{n=1}^{N} w_n(0)^{-4} < C$ uniformly in N. Moreover, $\frac{1}{N}\sum_{n\in E_N} w_n(0)$ has mean and variance bounded by

$$
\frac{p}{N}\sum_{n=1}^{N} w_n(0) \text{ and } \frac{(1/2)(1-1/2)}{N^2}\sum_{n=1}^{N} w_n(0)^2
$$

so it follows that $B_N(2) \to 0$ as $N \to \infty$.

Hence, we can bound (5.30) by

$$
\begin{aligned}
& B_N(2) + C\int_0^t \left[E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} |Q^N(\tau) - Q(\tau)|\right) + \frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\leq\tau\leq s\wedge\rho^N} E|W_n^N(\tau) - W_n(\tau)|\right)\right] ds \\
\leq\; & B_N(2) + C\int_0^t ED_N(s)ds.
\end{aligned}
$$

Putting together (5.29) and (5.30) we get

$$
\begin{aligned}
& \frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\leq\tau\leq t\wedge\rho^N} |W_n^N(\tau) - W_n(\tau)|\right) \\
\leq\; & B_N(2) + C\int_0^t \left[E\left(\sup_{0\leq\tau\leq t\wedge\rho^N} |Q^N(\tau) - Q(\tau)|\right) + \frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\leq\tau\leq t\wedge\rho^N} |W_n^N(\tau) - W_n(\tau)|\right)\right] ds.
\end{aligned}
$$

or

$$||\mathbf{W}^N(t) - \mathbf{W}(t)|| \leq B_N(2) + C\int_0^t D_N(s)ds.$$

Finally add in (5.29) and we get our Gronwall inequality. $D_N(t) \leq B_N + C\int_0^t D_N(s)ds$ where $B_N = B_N(1) + B_N(2)$.

5.2 Convergence at q_{max}

In the last section we have shown convergence in the interior. Now consider a path where $Q(t) = q_{max}$ on some interval $[0, \rho]$ which we may take to be inside $[0, T]$. On this interval then $K(t)$ satisfies

$$\overline{S}(t)(1 - K(t)) = \sum_{c=1}^{d} \kappa^c \langle Id, M_c(s)\rangle \frac{(1 - K(t))}{R_c(t)} \quad = \quad L$$

and $K(t) > p_{max}$. Note this means

$$L/\overline{S}(t) \leq 1 - p_{max}. \tag{5.36}$$

We assume $Q^N(0) = q_{max}$ and for $[0, \rho^N]$ $K^N(t)$ is given by

$$\overline{S}_N^N(t)(1 - K^N(t) = \sum_{c=1}^{d} \kappa_c^N \langle Id, M_c^N(t)\rangle \frac{(1 - K^N(t)}{R_c^N(t)} \quad = \quad L$$

with $K^N(t) > p_{max}$. Note this means

$$L/\overline{S}_N^N(t) \leq 1 - p_{max}. \tag{5.37}$$

We wish to show convergence on $[0, \rho \wedge \rho^N]$. We can repeat the steps in the last subsection but since $Q(t) = Q^N(t) = q_{max}$ (and consequently $R_n(t) = R^N(t)$) we need only consider

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N} E\left(\sup_{0\leq\tau\leq t\wedge\rho^N} |W_n^N(\tau) - W_n(\tau)|\right) \\
&\leq \; C\int_0^t ||W_n^N(s) - W_n(s)||ds \text{ from (5.29} \\
&\qquad + \frac{1}{N}\sum_{n=1}^{N} CE\left(\int_0^{t\wedge\rho^N} |\lambda_n^N(s) - \lambda_n(s))|ds\right) \text{ from (5.30)}
\end{aligned}$$

The bound on $E\left(\int_0^{t\wedge\rho^N}|\lambda_n^N(s)-\lambda_n(s))|ds\right)$ proceeds as in the last section except that (5.32), (5.33) and (5.35) are zero and (5.34) is replaced by

$$\begin{aligned}|K^N(s-R_n(s))-K(s-R_n(s))| &= |L/\overline{S}_N^N(s)-L/\overline{S}(s)|\\ &\le \frac{(1-p_{max})^2}{L}|\overline{S}_N^N(s)-\overline{S}(s)|\\ &\le C(|\overline{S}_N^N(s)-\overline{S}_N(s)|+|\overline{S}_N(s)-\overline{S}(s)|)\end{aligned}$$

Taking expectations shows the fourth term is bounded by

$$\begin{aligned}& C\int_0^t\frac{1}{N}\sum_{n=1}^N E\left(\sup_{0\le\tau\le s\wedge\rho^N}|W_n^N(\tau)-W_n(\tau)|\right)ds+\int_0^t E|\overline{S}_N(s)-\overline{S}(s)|ds\\ = \quad & B_N+C\int_0^t||W_n^N(s)-W_n(s)||ds\end{aligned}$$

where $B_N=\int_0^t|\overline{S}_N(s)-\overline{S}(s)|ds\to 0$ as $N\to\infty$.

Adding all the pieces together we get

$$||W_n^N(t)-W_n(t)||\le B_N+C\int_0^t||W_n^N(s)-W_n(s)||ds$$

and this means $||W_n^N(t)-W_n(t)||\to 0$ as $N\to\infty$ and we have convergence. The convergence for a path at the lower boundary is essentially the same.

6 Mean-field stochastic differential equations

In this section we prove Theorem 3. We can reformulate (2.7) as in [1].

$$\begin{aligned}&\langle g,M_c^N(t)\rangle-\langle g,M_c^N(0)\rangle \qquad (6.38)\\ &= \frac{1}{\kappa_c^N N}\sum_{n=1}^N\int_0^t\Big[\frac{dg_c}{dw}(W_n^N(s))\frac{1}{R_c^N(s)}ds\\ &\qquad +\left(g_c(W_n^N(s^-)/2)-g_c(W_n^N(s^-))\right)dN_n(\Lambda_n(s))\Big]\chi\{n\in K_c\}\\ &= \frac{1}{\kappa_c^N N}\sum_{n=1}^N\chi\{n\in K_c\}\int_0^t\Big[\frac{dg_c}{dw}(W_n(s))\frac{1}{R_c^N(s)}ds+\left(g_c(W_n^N(s)/2)-g_c(W_n^N(s))\right)\\ &\quad\cdot(1-\dot{R}_c^N(s))\frac{W_n^N(s-R_c^N(s))}{R_c^N(s-R_c(s))}K^N(s-R_c^N(s))ds\Big]+\mathcal{E}_c^N(t)\end{aligned}$$

(6.39)

where $\mathcal{E}_c^N(t)$ is given by

$$\frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\chi\{n\in K_c\}\int_0^t (g_c(W_n^N(s^-)/2)-g^c(W_n(s^-)))dZ_n(\Lambda_n^N(s))$$

and

$$Z_n(t)-Z_n(0) := \int_0^t \left(dN_n(\Lambda_n^N(s))-(1-\dot{R}_c^N(s))\frac{W_n^N(s-R_n^N(s))}{R_c^N(s-R_c^N(s))}K^N(s-R_c^N(s))ds\right).$$

Hence,

$$\begin{aligned}
&\langle g_c, M_c^N(t)\rangle - \langle g_c, M_c^N(0)\rangle \\
= &\int_0^t \Big[\frac{1}{R_c^N(s)}\langle\frac{dg_c(w)}{dw}, M_c^N(s)\rangle \qquad (6.40)\\
&+\langle (g^c(w/2)-g^c(w))v, M_c^N(s-R_c^N(s),dv;s,dw)\rangle\frac{(1-\dot{R}_c^N(s))}{R_c^N(s-R_c^N(s))}K^N(s-R_c^N(s))\Big]ds\\
+ &\ \mathcal{E}_c^N(t)\\
= &\int_0^t \Big[\frac{1}{R_c^N(s)}\langle\frac{dg_c(w)}{dw}, M_c^N(s)\rangle ds \qquad (6.41)\\
&+\langle (g_c(w/2)-g_c(w))v, M_c^N(s-R_c^N(s),dv;s,dw)\rangle\frac{(1-\dot{R}_c^N(s))}{R_c^N(s-R_c^N(s))}K^N(s-R_c^N(s))ds\Big]\\
+ &\ \mathcal{E}_c^N(t).
\end{aligned}$$

We first show $\mathcal{E}_c^N$ is asymptotically small as $N\to\infty$. Recall

$$\mathcal{E}_c^N(t) \quad = \quad \frac{1}{\kappa_c^N N}\sum_{n=1}^{N}\int_0^t C_{c,n}^N(s)Z_{c,n}^N(ds)$$

where

$$C_{c,n}^N(s)=\chi\{n\in K_c\}\left(g(W_n^N(s)/2)-g(W_n^N(s))\right)$$

and

$$Z_{c,n}^N(t)-Z_{c,n}^N(0) := \int_0^t (dN_n(\Lambda_n^N(s))-(1-\dot{R}_c^N(s))\frac{W_n^N(s-R_c^N(s))}{R_c^N(s-R_c^N(s))}K^N(s-R_c^N(s))ds)\chi\{n\in K_c\}.$$

If $n\in K^c$, $N_n(\Lambda_n^N(s))$ is a point process adapted to to $\mathcal{F}_n(t)$ with a stochastic intensity $W_n^N(s-R_c^N(s))K^N(s-R_c^N(s))/R_c^N(s-R_c^N(s))$. Consequently $Z_n(t)$ is a martingale. Recall that $W_n^N(s)$ is also adapted to $\mathcal{F}_n(t)$ so the right continuous version is $\mathcal{F}_n(t)$-predictable. By Theorem T13 in [3]

$$
\begin{aligned}
& E\left(\mathcal{E}_N^c(t)\right)^2 \\
= & \frac{1}{(\kappa_N^c N)^2} E\left[\sum_{n=1}^{N} \chi\{n \in K_c\} \int_0^t (C_{n,c}^N)^2(s)(1-\dot{R}_c^N(s)) \frac{W_n^N(s-R_c^N(s))}{R_c^N(s-R_c^N(s))} K^N(s-R_c^N(s)) ds\right] \\
\leq & \frac{1}{(\kappa_N^c N)^2} E \sum_{n=1}^{N} \chi\{n \in K_c\} \int_0^t (g_c(W_n^N(s)/2) - g_c(W_n^N(s))^2 \frac{W_n^N(s-R_c^N(s))}{R_c^N(s-R_c^N(s))} ds\Big) \\
\leq & \frac{C_1}{(\kappa_N^c N)^2} \left(\sum_{n=1}^{N} \chi\{n \in K_c\} \int_0^t E\left[W_n^N(s-R_c^N(s))\right] ds\right)
\end{aligned}
$$

where C_1 is a constant depending on $\sup g_c$ and $\sup (g_c)'$.

We have the apriori bound $W_n^N(t) \leq a(t)$. We also have an apriori bound on $E(1/W_n^N(t))$ if we ignore the positive drift and focus on the number of jumps down by half in time t. The probability of k jumps down to $w_n(0)/2^k$ is bounded above by the probability a Poisson random variable with parameter $\lambda = (w_n(0)+t/T_{min})/T_{min}$ takes the value k (since $W_n^N(s-R_c^N(s))K^N(s-R_c^N(s))/R_c^N(s-R_c^N(s)) \leq (w_n(0)+t/T_{min})/T_{min}$ for $s \leq t$ and we assume $W_n^N(s-R_c^N(s)) = w_n(0)$ for $s \leq R_c^N(s)$). Hence

$$
E(1/W_n^N(t)) \leq \sum_{k=0}^{\infty} \frac{2^k}{w_n(0)} \exp(-\lambda) \frac{\lambda^k}{k!} = C_2
$$

where $C_2 = 2\lambda/w_n(0) = 2 + 2t/(T_{min}^2 w_n(0))$.

From the above we conclude

$$
E\left(\mathcal{E}_c^N(t)\right)^2 \leq \frac{tC_1}{(\kappa_c^N)^2 N} (\left(2 + \frac{2t}{T_{min}^2} \frac{1}{N} \sum_{n=1}^{N} \frac{1}{w_n(0)}\right) + \left(t/T_{min} + \frac{1}{(\kappa_c^N)^2 N} \sum_{n=1}^{N} w_n(0)\right)).
$$

So, by Assumptions 1 and 2, $\mathcal{E}_c^N(t)$ tends to 0 in L^2. Since, in addition, $\mathcal{E}_c^N(t)$ is a martingale it follows that

$$
P(\sup_{t\in[0,T]} |\mathcal{E}_c^N(t)| > \lambda) \leq E\left(\mathcal{E}_c^N(T)\right)^2 / \lambda^2
$$

so the process $\mathcal{E}_c^N(t), t \in [0,T]$ converges to zero almost surely.

The processes $W_n^N(t)$, $Q^N(t)$ and $K^N(t)$ converge in the Skorohod topology to the limit processes $Q(t)$, $K(t)$ while $(M_1^N(t), \ldots M_d^N(t))$ converges weakly to $(M_1(t), \ldots M_d(t))$ where the limit processes satisfy (3.18) and (3.17). Take the limit of (6.41) and we have our proof.

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